Introduction

The Roosevelt School District values the importance of developing a strong foundation in mathematics to support students with developing the capacity to think and reason mathematically. As outlined by the National Council for Teachers of Mathematics, the following represents the expectations we have in the Roosevelt School District for our students and teachers:

- Every student deserves an excellent program of instruction in mathematics that challenges each student to achieve at the high level required to become college and career ready.
- Every student must be taught by teachers who have a sound knowledge of mathematics and how children learn mathematics.
- Every student must be taught by teachers who hold high expectations for themselves and their students.
- Teachers guide the learning process in their classrooms and manage the classroom environment through a variety of instructional approaches directly tied to the mathematics content and to students’ needs.
- Students use diverse strategies and different algorithms to solve problems, and teachers must recognize and take advantage of these alternative approaches to help students develop a better understanding of mathematics.
- Computational skills and number concepts are essential components of the mathematics curriculum, and a knowledge of estimation and mental computation are more important than ever. By the end of the middle grades, students should have a solid foundation in number, algebra, geometry, measurement, and statistics.
- Learning mathematics is maximized when teachers focus on mathematical thinking and reasoning. Progressively more formal reasoning and mathematical proof should be integrated into the mathematics program as a student continues in school.
- Learning mathematics is enhanced when content is placed in context and is connected to other subject areas and when students are given multiple opportunities to apply mathematics in meaningful ways as part of the learning process.

This handbook has been designed by the Curriculum and Assessment Department for the teachers of Roosevelt School District to provide support and guidance with all aspects of mathematics. Please let us know if you have any questions or if you need additional guidance and support with your math instruction.

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Instructional Shifts

The Arizona College and Career Ready Standards in Mathematics (AZCCRSM) require three (3) key instructional shifts in order to support our students achieving high levels of proficiency in mathematics.

Greater **focus** on fewer topics

The standards call for greater focus in mathematics. Rather than racing to cover many topics in a mile-wide, inch-deep curriculum, the standards ask math teachers to significantly narrow and deepen the way time and energy are spent in the classroom. This means focusing deeply on the major work of each grade as follows:

- In grades K–2: Concepts, skills, and problem solving related to addition and subtraction
- In grades 3–5: Concepts, skills, and problem solving related to multiplication and division of whole numbers and fractions
- In grade 6: Ratios and proportional relationships, and early algebraic expressions and equations
- In grade 7: Ratios and proportional relationships, and arithmetic of rational numbers
- In grade 8: Linear algebra and linear functions

This focus will help students gain strong foundations, including a solid understanding of concepts, a high degree of procedural skill and fluency, and the ability to apply the math they know to solve problems inside and outside the classroom.
**Coherence**: Linking topics and thinking across grades

Mathematics is not a list of disconnected topics, tricks, or mnemonics; it is a coherent body of knowledge made up of interconnected concepts. Therefore, the standards are designed around coherent **progressions** from grade to grade.

Learning is carefully connected across grades so that students can build new understanding onto foundations built in previous years. For example, in 4th grade, students must “apply and extend previous understandings of multiplication to multiply a fraction by a whole number” (Standard 4.NF.4). This extends to 5th grade, when students are expected to build on that skill to “apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction” (Standard 5.NF.4). Each standard is not a new event, but an extension of previous learning.

Coherence is also built into the standards in how they reinforce a major topic in a grade by utilizing supporting, complementary topics. For example, instead of presenting the topic of data displays as an end in itself, the topic is used to support grade-level word problems in which students apply mathematical skills to solve problems.

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**K-8 Domains Progression**

<table>
<thead>
<tr>
<th>Domains</th>
<th>K</th>
<th>1</th>
<th>2</th>
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<td><strong>Counting and Cardinality</strong></td>
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</tbody>
</table>
**Rigor**: Pursue conceptual understanding, procedural skills and fluency, and application with equal intensity

Rigor refers to deep, authentic command of mathematical concepts, not making math harder or introducing topics at earlier grades. To help students meet the standards, educators will need to pursue, with equal intensity, three aspects of rigor in the major work of each grade: conceptual understanding, procedural skills and fluency, and application.

**Conceptual understanding**

The standards call for conceptual understanding of key concepts, such as place value and ratios. Students must be able to access concepts from a number of perspectives in order to see math as more than a set of mnemonics or discrete procedures.

**Procedural skills and fluency**

The standards call for speed and accuracy in calculation. Students must practice core functions, such as single-digit multiplication, in order to have access to more complex concepts and procedures. Fluency must be addressed in the classroom or through supporting materials, as some students might require more practice than others.

**Application**

The standards call for students to use math in situations that require mathematical knowledge. Correctly applying mathematical knowledge depends on students having a solid conceptual understanding and procedural fluency.

Source: [http://www.corestandards.org/other-resources/key-shifts-in-mathematics/](http://www.corestandards.org/other-resources/key-shifts-in-mathematics/)

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![Diagram](https://via.placeholder.com/150)

**Rigor through a Balanced Approach**

- Conceptual Understanding
- Application
- Procedural Fluency (Efficiency, Accuracy, Flexibility)
Balanced Math Framework

This daily framework is intended to ensure an appropriate balance between application, procedural fluency, and developing conceptual understanding within a 90-minute math block.

Adjustments may need to be made for classes with less than the recommended time allotment.

<table>
<thead>
<tr>
<th>Daily Math Framework</th>
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</thead>
<tbody>
<tr>
<td><strong>MathReview (10 minutes)</strong></td>
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<tr>
<td>Skill review:</td>
</tr>
<tr>
<td>• Share 3-5 problems a day with students</td>
</tr>
<tr>
<td>• Students solve problems in their notebooks or math journals.</td>
</tr>
<tr>
<td>• Five minutes of work time and five minutes to correct.</td>
</tr>
<tr>
<td>• Correct together and have students share the various ways they solved the problem.</td>
</tr>
</tbody>
</table>

| Mental Math and/or Fact Fluency (10 minutes) |
| Works to develop students’ mental mathematical abilities: |
| • Read a number problem aloud for students (should be developmentally appropriate). |
| • Students solve mentally. |
| • Students should give the correct answer (or show on a white board) for a quick check. |
| Build math fact automaticity: |
| • Have students work at their independent level practicing math facts. |

| Concept Lesson (30-40 minutes) |
| Instructional Approach = Construct Knowledge and/or Explicit Modeling |
| Helps students develop a clear conceptual understanding of mathematics: |
| • Problem-based interactive learning should be the foundation in teaching for understanding. |
| • Provide the focus of the lesson by sharing the purpose of the lesson. |
| • Use multiple methods and strategies. |
| • Incorporate concrete models that support the understanding of mathematical concepts. |
| • Provide a variety of instructional opportunities from whole class to partners and small group activities. |
| • Make connections to aid students in the application of the mathematical knowledge. |
| • Provide opportunities for students to discover concepts using hands-on or problem–based learning activities. |

| Closure (5-10 minutes) |
| Provides a way to check student understanding: |
| • Provide time for students to share prior knowledge, reflect on new learning, and make connections. |
| • Students articulate their thinking (this can be done verbally or in writing, including pictures and words). |
| • Use formative assessment as a post-assessment or performance task to check for understanding. |

| Small group, centers, assessments or problem-based activities (20-30 minutes) |
| Allows for students to be given time to receive additional instruction, remediation or enrichment opportunities: |
| • Place students in differentiated instruction groups (based on assessment information gathered throughout the week). |
| • Students in need of remediation should be grouped together and receive direct, explicit instruction from teacher. |
| Helps students learn how to mathematically communicate how to solve authentic complex problems: |
| • Provide developmentally appropriate activities. |
| • Make intentional connections to the concepts being taught. |
| • Make sure the students understand the expectations of the activity. |
| • Emphasize how the problem was solved, what strategies were used, and how the answer will be shared. |
Problem-Based Learning

Problem-based learning (PBL) describes an environment where ‘problems’ drive learning, or in other words learning begins when there is a problem to be solved and the learner must gain new knowledge in order to solve that problem.

Learning is therefore driven by problematic mathematics rather than by the memorization of facts, formulas, and procedures. Students no longer seek single answers, but they instead gather information, pose and identify different solution methods, evaluate their options, and then present a solution.

The ultimate goal of math education is to promote understanding and transfer; students understand mathematics when they invent and examine their own solutions for solving mathematical problems, which is what PBL strives to achieve. Thus, "problem-based learning is a classroom strategy that organizes mathematics instruction around problem solving activities and affords students more opportunities to think critically, present their own creative ideas, and communicate with peers mathematicially" (Roh, 2003, p. 2).

In the realm of mathematics, students have an increased understanding of mathematical concepts, word problems, and planning capabilities. They also acquire positive attitudes toward mathematics in general and the teacher’s feedback. Thus, students gain an in-depth understanding of mathematics in a PBL environment. This approach allows students to change and adapt their thinking and methods to new situations. There is no longer a cookbook style recipe to follow where rules, exercises, formulas, and procedures occupy precious classroom time. Students in PBL environments have the opportunity to learn mathematical processes and skills that are associated with communication, representation, modeling, and reasoning.

http://tccl.rit.albany.edu/
Problem Solving Implementation Guide

Selecting a Task - Things to Consider
- The level of cognitive demand
- The mathematics students will apply and learn
- The accessibility of the task to students

Planning for Implementation - Anticipate Student Responses
- Possible student misconceptions
- Different strategies and tools students might use
- Language that indicates understanding

Planning for Implementation - Determine Questions to Ask
- To surface and clarify misconceptions
- To probe students’ understanding
- To assess and advance student reasoning
- To make the mathematics visible
- To encourage reflection and justification

Implementing the Task
- Teacher presents the problem/task, facilitates the KFA process and does not model or suggest a solution process.
- Teacher has tools available for student use.
- Students first work individually, then in pairs/groups while the teacher:
  - records data on students’ engagement in the Standards for Mathematical Practice (SMP),
  - asks guiding questions that promote the SMP and productive struggle,
  - assesses students’ understanding, and
  - chooses students to present and determines the sequence of presentations that will allow all students to have access to the mathematical thinking and different representations (Note that sequencing presentations from a concrete to abstract thinking and less sophisticated to more sophisticated strategies works best). The teacher must select presenters to ensure that the mathematics that is at the heart of the lesson actually gets on the table.

Discussing the Task
- Teacher facilitates discourse requiring students to explain, defend, ask questions, clarify, model with equations, use representations and appropriate tools, use precise language, make connections, etc.
- Teacher facilitates a class summary and an individual reflection.
When presenting problems or tasks to students, it should become second nature for students to follow a simple K-F-A process to persevere in solving the problem (SMP1) and construct viable arguments (SMP3). Students should reflect upon or consider the following questions prior to tackling the calculations.

**K** What do you know about the situation? What’s going on?

**F** What do you need to find out? What will be the answer statement? Use the units.

**A** What do you know about the answer? What is a good estimate of the answer? Use <, >, or about.

Nora Ramirez, Math Consultant
## Common Addition and Subtraction Situations*

<table>
<thead>
<tr>
<th>Add To</th>
<th>Change Unknown</th>
<th>Start Unknown</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now? (2 + 3 = ?)</td>
<td>Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were five bunnies. How many bunnies hopped over to the first two? (2 + ? = 5)</td>
<td>Some bunnies were sitting on the grass. Three more bunnies hopped there. How many bunnies were on the grass before? (? + 3 = 5)</td>
</tr>
<tr>
<td>Take From</td>
<td></td>
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<tr>
<td>Five apples were on the table. I ate two apples. How many apples are on the table now? (5 - 2 = ?)</td>
<td>Five apples were on the table. I ate some apples. Then there were three apples. How many apples did I eat? (5 - ? = 3)</td>
<td>Some apples were on the table. I ate two apples. Then there were three apples. How many apples were on the table before? (? - 2 = 3)</td>
</tr>
<tr>
<td>Total Unknown</td>
<td>Both Addends Unknown(^1)</td>
<td>Addend Unknown(^2)</td>
</tr>
<tr>
<td>Three red apples and two green apples are on the table. How many apples are on the table? (3 + 2 = ?)</td>
<td>Grandma has five flowers. How many can she put in her red vase and how many in her blue vase? (5 = 0 + 5, 5 = 5 + 0, 5 = 1 + 4, 5 = 4 + 1, 5 = 2 + 3, 5 = 3 + 2)</td>
<td>Five apples are on the table. Three are red and the rest are green. How many apples are green? (3 + ? = 5, 5 - 3 = ?)</td>
</tr>
<tr>
<td>Put Together / Take Apart</td>
<td></td>
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<tr>
<td>Compare</td>
<td>Bigger Unknown</td>
<td>Smaller Unknown</td>
</tr>
<tr>
<td>&quot;How many more?&quot; version: Lucy has two apples. Julie has five apples. How many more apples does Julie have than Lucy? (2 + ? = 5, 5 - 2 = ?)</td>
<td>&quot;More&quot; version suggests operation: Julie has three more apples than Lucy. Lucy has two apples. How many apples does Julie have? (2 + 3 = ?, 3 + 2 = ?)</td>
<td>&quot;Fewer&quot; version suggests operation: Lucy has 3 fewer apples than Julie. Julie has five apples. How many apples does Lucy have?</td>
</tr>
<tr>
<td>&quot;How many fewer?&quot; version: Lucy has two apples. Julie has five apples. How many fewer apples does Lucy have than Julie? (2 + ? = 5, 5 - 2 = ?)</td>
<td>&quot;Fewer&quot; version suggests operation: Lucy has 3 fewer apples than Julie. Lucy has two apples. How many apples does Julie have? (2 + 3 = ?, 3 + 2 = ?)</td>
<td>&quot;More&quot; version suggests operation: Julie has three more apples than Lucy. Julie has five apples. How many apples does Lucy have? (5 - 3 = ?, ? + 3 = 5)</td>
</tr>
</tbody>
</table>

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*Adapted from Mathematics Learning in Early Childhood, National Research Council, AZ Mathematics Standards, and Progressions for the CCSS in Mathematics

Darker shading indicates the four Kindergarten problem subtypes. Grade 1 and 2 students work with all subtypes. Unshaded (white) problems are the four difficult subtypes that students should work with in Grade 1 but need not master until Grade 2.

\(^1\)These take apart situations can be used to show all the decompositions of a given number. The associated equations, which have the total on the left of the equal sign, help children understand that the = sign does not always mean makes or results in but always does mean is the same number as.

\(^2\)Either addend can be unknown, so there are three variations of these problem situations. Both Addends Unknown is a productive extension of this basic situation, especially for small numbers less than or equal to 10.
## Common Multiplication and Division Situations

<table>
<thead>
<tr>
<th>Unknown Product</th>
<th>Group Size Unknown</th>
<th>Number of Groups Unknown</th>
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</thead>
<tbody>
<tr>
<td>$3 \times 6 = ?$</td>
<td>$3 \times ? = 18$, and $18 ÷ 3 = ?$</td>
<td>? $\times 6 = 18$, and $18 ÷ 6 = ?$</td>
</tr>
</tbody>
</table>

#### 3rd Grade

**Equal Groups**
- There are 3 bags with 6 plums in each bag. How many plums are there in all? *Measurement example.*
- You need 3 lengths of string, each 6 inches long. How much string will you need altogether? *Measurement example.*

**Arrays**
- There are 3 rows of apples with 6 apples in each row. How many apples are there? *Area example.*
- What is the area of a 3 cm by 6 cm rectangle? *Area example.*

#### 4th Grade

**Compare**
- A blue hat costs $6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost? *Measurement example.*
- A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be 3 times as long? *Measurement example.*

**General**
- $a \times b = ?$ *$a \times ? = p$, and $p ÷ a = ?$* *$? \times b = p$, and $p ÷ b = ?$*

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7The first examples in each cell are examples of discrete things. These are easier for students and should be given before the measurement examples.

4The language in the array examples shows the easiest form of array problems. A harder form is to use the terms rows and columns: The apples in the grocery window are in 3 rows and 6 columns. How many apples are in there? Both forms are valuable.

5Area involves arrays of squares that have been pushed together so that there are no gaps or overlaps, so array problems include these especially important measurement situations.
Concrete-Pictorial-Abstract (CPA) Approach

Research has shown that the optimal presentation sequence to teach new mathematical content is through the concrete-pictorial-abstract (CPA) approach (Sousa, 2008). This approach also goes by other names: the concrete-representational-abstract approach or the concrete-semiconcrete-abstract approach. Regardless of the terminology used, the instructional approach is similar and is based on the work of Jerome Bruner (Bruner, 1960). Through this approach, students are experiencing and discovering mathematics rather than simply regurgitating it.

- **Concrete.** At the concrete level, tangible objects, such as manipulatives, are used to approach and solve problems. Examples of concrete tools include: unifix cubes, Cuisenaire rods, fraction circles and strips, base-10 blocks, double-sided foam counters, or measuring tools. Almost anything students can touch and manipulate to help approach and solve a problem is used at the concrete level.

- **Pictorial.** At the pictorial level, representations are used to approach and solve problems. These can include drawings (e.g., circles to represent coins, pictures of objects, tally marks, number lines), diagrams, charts, and graphs. These pictures are visual representations of the concrete manipulatives. It is important for the teacher to explain this connection.

- **Abstract.** At the abstract level, symbolic representations are used to approach and solve problems. These representations can include numbers or letters. It is important for teachers to explain how symbols can provide a shorter and efficient way to represent numerical operations.

Joan Gujarati, Ed.D.
Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important “processes and proficiencies” with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council’s report Adding It Up: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy).

Grouping the Eight Mathematical Practices

Reasoning and Explaining

2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others.

Modeling and Using Tools

4. Model with mathematics
5. Use appropriate tools strategically

Seeing Structure and Generalizing

7. Look for and make use of structure
8. Look for and express regularity of repeated reasoning
<table>
<thead>
<tr>
<th>Mathematical Practice</th>
<th>Student Actions</th>
<th>Teacher Actions</th>
<th>Related Questions</th>
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</table>
| 1. Make sense of problems and persevere in solving them | **Have or value sense-making**  
**Use patience and persistence to listen to others**  
**Be able to use and make sense of strategies**  
**Monitor progress and change course, if needed**  
**Be able to show, use, and explain representations and use them to solve problems**  
**Communicate, verbally and in written format**  
**Be able to deduce what is a reasonable solution in the context of the problem** | **Provide open-ended and rich problems**  
**Ask probing questions**  
**Model multiple problem-solving strategies through Think-Alouds**  
**Promote and value discourse, collaboration, and student presentations**  
**Provide cross-curricular integrations**  
**Probe student responses (correct or incorrect) for understanding of approaches**  
**Provide solutions** | **How would you describe the situation in your own words?**  
**How would you describe what you are trying to find?**  
**What diagram or manipulatives can you use to make sense of what you need to do?**  
**What information is given in the problem?**  
**What is the relationship between the quantities?**  
**Describe what you have already tried. What might you change?**  
**Talk through the steps you’ve used to this point.**  
**What steps in the process are you most confident about?**  
**What are some other strategies you might try?**  
**How might you use one of your previous problems to help you begin?**  
**How else might you organize...represent...show...?** |
| 2. Reason abstractly and quantitatively | **Make sense of and explain quantities and relationships in problem situations**  
**Create and explain multiple representations**  
**Create and explain equivalent expressions or equations**  
**Use context to reason about an operation, an answer or the units of the answer**  
**Translate from abstract to context & vice versa**  
**Estimate first/check if answer reasonable**  
**Make connections**  
**Consider whether strategies are efficient**  
**Take time and make effort to reason** | **Develop opportunities for problem solving**  
**Provide opportunities for students to listen to the reasoning of other students**  
**Give time for processing and discussing**  
**Tie content areas together to help make connections**  
**Ask students to explain their reasoning**  
**Think aloud for student benefit**  
**Value the path to developing efficient strategies**  
**Emphasize reasoning, not just answer getting** | **What do the numbers used in the problem represent?**  
**What is the relationship of the quantities?**  
**What is a reasonable answer to this problem? How do you think about that?**  
**How is _____ related to _____?**  
**What is the relationship between _____ and _____?**  
**What does _____ mean to you? (e.g. symbol, quantity, diagram)**  
**What properties might we use to find a solution?**  
**How did you decide in this task that you needed to use...?**  
**Could we have used another operation or property to solve this task? Why/why not?**  
**Why does that make sense?** |

Nora G. Ramirez  
Adapted from: Actions and dispositions from NCSM Summer Leadership Academy, Atlanta, GA • Draft, June 22, 2011  
Questions adapted from Common Core State Standards Flip Book
The Standards for Mathematical Practice: 2nd-8th Grade Student and Teacher Actions and Related Questions

<table>
<thead>
<tr>
<th>Mathematical Practice</th>
<th>Student Actions</th>
<th>Teacher Actions</th>
<th>Related Questions</th>
</tr>
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</table>
| 3. **Construct viable arguments and critique the reasoning of others** | • Ask questions of students and teacher  
• Justify and communicate predictions and conclusions  
• Use examples and non-examples  
• Analyze data, use to make arguments  
• Use objects, drawings, diagrams, and actions  
• Use mathematics vocabulary, properties, and definitions in support of statements  
• Listen and respond to others  
• Build on other students’ ideas  
• Question and comment on other’s work/ideas | • Create a safe environment for risk-taking and critiquing with respect  
• Model each key student disposition  
• Provide complex, rigorous tasks that foster deep thinking  
• Provide time for student presentations and student-to-student discourse  
• Plan effective questions and student grouping  
• Ask students to agree, disagree, support and compare the ideas of others | • What mathematical evidence would support your solution?  
• How can we be sure that...? / How could you prove that...?  
• Will it still work if...?  
• What were you considering when...?  
• How did you decide to try that strategy?  
• How did you test whether your approach worked?  
• How did you decide what the problem was asking you to find?  
• Did you try a method that did not work? Why didn’t it work? Could it work?  
• What is the same and what is different about...?  
• How could you demonstrate a counter-example? |
| 4. **Model with mathematics** | • Use mathematics (numbers and symbols) to solve/work out real-life situations  
• Mathematize situations using numbers, symbols, equations, tables, graphs, or formulas  
• Pull out important information needed to solve a problem when approached with several factors in everyday situations  
• Make sense of the symbols and quantities in an equation or function (as they relate to the context) | • Allow time for the process to take place (equations, graphs, etc.)  
• Stress the importance of connecting the context, equations, tables and/or graphs  
• Emphasize sense making between a context, symbols and quantities in an equation  
• Provide meaningful, real world, authentic, performance-based tasks (nontraditional word problems) | • How can you use numbers to represent the problem?  
• What are some other ways to represent the quantities?  
• What is an equation or expression that matches the diagram, number line, chart, table, or your actions with the manipulatives? Is there more than one equation?  
• Where did you see one of the quantities in the task in your equation or expression? What does each number in the equation mean?  
• How would it help to create a diagram, graph, table...?  
• What are some ways to visually represent...?  
• What formula might apply in this situation? |
<table>
<thead>
<tr>
<th>Mathematical Practice</th>
<th>Student Actions</th>
<th>Teacher Actions</th>
<th>Related Questions</th>
</tr>
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</table>
| 5. Use appropriate tools strategically | • Choose the appropriate tool to solve a given problem and deepen conceptual understanding (paper/pencil, ruler, base 10 blocks, compass, protractor)  
• Choose the appropriate technological tool to solve a given problem and deepen conceptual understanding (e.g., spreadsheet, geometry software, calculator, web 2.0 tools)  
• Use technology to explore mathematical situations  
• Know to examine answers from calculators or software programs for reasonableness | • Maintain knowledge of appropriate tools  
• Make tools available for student selection  
• Model use of the tools available, their benefits and limitations  
• Scaffold the understanding and use of more complex tools  
• Model a situation where the decision needs to be made as to which tool should be used  
• Provide tasks that require students to use manipulatives, calculators or software programs to develop conceptual understanding, solve problems, or predict solutions. | • What mathematical tools can we use to visualize and represent the situation?  
• Which tool is more efficient? Why do you think so?  
• What does (a manipulative) represent?  
• How can (the tool) help you understand the situation/estimate the answer/find a solution?  
• In this situation would it be helpful to use a graph, a number line, a ruler, a diagram, a calculator, or a manipulative?  
• Why was it helpful to use...?  
• What mathematical terms apply in this situation?  
• How did you know your solution was correct?  
• What could you test your solution to see if it answers the problem?  
• When you said _____, what did you mean? |
| 6. Attend to precision | • Communicate with precision—orally and written  
• Use mathematics concepts and vocabulary appropriately.  
• State meaning of symbols; use appropriately  
• Attend to units/labeling/tools accurately  
• Carefully formulate explanations  
• Calculate accurately and efficiently  
• Express answers in terms of context  
• Formulate precise definitions with others  
• Use journals or class charts as reference | • Model Think aloud/Talk aloud  
• Give explicit instruction through the use of think aloud/talk aloud  
• Guide inquiry: teacher gives problem, students work together to solve problems, and time is given for discussing/sharing/comparing  
• Ask probing questions related to the content  
• Ask for more specificity about an explanation  
• Have materials available for students to use as reference (journals, charts, books, etc.) | • What mathematical terms apply in this situation?  
• How did you know your solution was correct?  
• Explain how you might show that your solution answers the problem.  
• How are you showing the meaning of the quantities?  
• What symbols or mathematical notations are important in this problem?  
• What mathematical language..., definitions..., properties can you use to explain...?  
• How could you test your solution to see if it answers the problem?  
• When you said _____, what did you mean? |
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<tr>
<td>7. Look for and make use of structure</td>
<td>• Look for, interpret, and identify patterns and structures&lt;br&gt;• Make connections to skills and strategies previously learned to solve new problems/tasks&lt;br&gt;• Reflect and recognize various structures in mathematics&lt;br&gt;• Breakdown complex problems into simpler, more manageable chunks</td>
<td>• Be quiet and allow students to think aloud&lt;br&gt;• Facilitate learning by using open-ended questioning to assist students in exploration&lt;br&gt;• Carefully select tasks that allow for students to make connections&lt;br&gt;• Allow time for student discussion and processing&lt;br&gt;• Foster persistence/stamina in problem solving&lt;br&gt;• Provide graphic organizers or record student responses strategically to allow students to discover patterns</td>
<td>• What observations do you make about...?&lt;br&gt;• What do you notice when...?&lt;br&gt;• What parts of the problem might you eliminate..., simplify...?&lt;br&gt;• What patterns do you find in...?&lt;br&gt;• How do you know if something is a pattern?&lt;br&gt;• What ideas that we have learned before were useful in solving this problem?&lt;br&gt;• What are some other problems that are similar to this one?&lt;br&gt;• How does this relate to...?&lt;br&gt;• In what ways does this problem connect to other mathematical concepts?</td>
</tr>
<tr>
<td>8. Look for and express regularity in repeated reasoning</td>
<td>• Identify patterns and make generalizations&lt;br&gt;• Continually evaluate reasonableness of intermediate results&lt;br&gt;• Maintain oversight of the process</td>
<td>• Provide rich and varied tasks that allow students to generalize relationships and methods, and build on prior math knowledge&lt;br&gt;• Provide adequate time for exploration&lt;br&gt;• Provide time for dialogue and reflection&lt;br&gt;• Ask deliberate questions that enable students to reflect on their own thinking&lt;br&gt;• Create strategic and intentional check-in points during student work time</td>
<td>• Explain how this strategy works in other situations?&lt;br&gt;• Is this always true, sometimes true or never true?&lt;br&gt;• How would we prove that...?&lt;br&gt;• What do you notice about...?&lt;br&gt;• What is happening in this situation?&lt;br&gt;• What would happen if...?&lt;br&gt;• Is there a mathematical rule for...?&lt;br&gt;• What predictions or generalizations can this pattern support?&lt;br&gt;• What mathematical consistencies do you notice?</td>
</tr>
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## The Standards for Mathematical Practice: K-1 Student and Teacher Actions and Related Questions

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<tbody>
<tr>
<td><strong>1. Make sense of problems and persevere in solving them</strong></td>
<td>Have or value sense-making Use patience and persistence to listen to others Be able to use strategies Use self-evaluation and redirections Be able to show or use multiple representations Communicate both verbally and in written format Be able to deduce what is a reasonable solution</td>
<td>Provide open-ended and rich problems Ask probing questions Model multiple problem-solving strategies through Think-Alouds Promote and value discourse and collaboration Cross-curricular integrations Probe student responses (correct or incorrect) for understanding and multiple approaches Provide solutions</td>
<td>Can you tell us about the problem? What question are you trying to answer? What do you notice about? What does the problem tell us? What do you know about the quantities, the numbers, etc.? Does the answer make sense? What can you do differently? Can you show us another way?</td>
</tr>
<tr>
<td><strong>2. Reason abstractly and quantitatively</strong></td>
<td>Create multiple representations Interpret problems in contexts Estimate first/answer reasonable Make connections Represent symbolically Visualize problems Talk about problems, real life situations Attend to units Use context to think about a problem</td>
<td>Develop opportunities for problem solving Provide opportunities for students to listen to the reasoning of other students Give time for processing and discussing Tie content areas together to help make connections Give real world situations Think aloud for student benefit Value invented strategies and representations Less emphasis on the answer</td>
<td>What do the numbers in this problem mean? What do you know about the answer? (emphasize units) (Looking at an equation) What do know about the ____ and the ____ ? Why did you add? Why did you subtract? How do you know that they are the same/equal? Could you have written the equation another way? Could you have written a different equation?</td>
</tr>
</tbody>
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Questions from: Nora Ramirez and Galveston School kindergarten teachers with input from the Shumway School 1st grade teachers (Chandler USD) April 11, 2013; Actions and dispositions from NCSM Summer Leadership Academy, Atlanta, GA • Draft, June 22, 2011
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<td>3. Construct viable arguments and critique the reasoning of others</td>
<td>Ask questions Use examples and non-examples Analyze data Use objects, drawings, diagrams, and actions Students develop ideas about mathematics and support their reasoning Listen and respond to others Encourage the use of mathematics vocabulary</td>
<td>Create a safe environment for risk-taking and critiquing with respect Model each key student disposition Provide complex, rigorous tasks that foster deep thinking Provide time for student discourse Plan effective questions and student grouping</td>
<td>Can you show us how you know? Can you prove it? Will it still work if...? What were you thinking when? Why did you use that strategy? How do you know that your strategy worked? Problem: K- What do you know is happening? F- What do you want to find? A- What do you know about the answer? What is the same and what is different about . . .?</td>
</tr>
<tr>
<td>4. Model with mathematics</td>
<td>Realize they use mathematics (numbers and symbols) to solve/work out real-life situations When approached with several factors in everyday situations, be able to pull out important information needed to solve a problem. Show evidence that they can use their mathematical results to think about a problem and determine if the results are reasonable. If not, go back and look for more information Make sense of the mathematics</td>
<td>Allow time for the process to take place (model, make graphs, etc.) Model desired behaviors (think alouds) and thought processes (questioning, revision, reflection/written) Make appropriate tools available Create an emotionally safe environment where risk taking is valued Provide meaningful, real world, authentic, performance-based tasks (nontraditional work problems)</td>
<td>How can you use manipulatives, pictures or numbers to show . . . what the problem means? Explain why your work makes sense. How can you use your picture to show how you got your answer? How can you show this with an equation? How does the equation match your story problem?</td>
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<td>5. Use appropriate tools strategically</td>
<td>Choose the appropriate tool to solve a given problem and deepen their conceptual understanding (paper/pencil, ruler, base 10 blocks, compass, protractor) Choose the appropriate technological tool to solve a given problem and deepen their conceptual understanding (e.g., spreadsheet, geometry software, calculator, web 2.0 tools)</td>
<td>Maintain appropriate knowledge of appropriate tools Effective modeling of the tools available, their benefits and limitations Model a situation where the decision needs to be made as to which tool should be used</td>
<td>What did you use to show the problem? Why did you choose that tool? Could you have used something else? Note: Ask questions to promote students thinking about choosing tools that are efficient. (I noticed that it took a long time to draw a circle to show each of those 38 bears. Is there a way you could have done this in less time?)</td>
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<td>6. Attend to precision</td>
<td>Communicate with precision—orally and written Use mathematics concepts and vocabulary appropriately State meaning of symbols and use appropriately Attend to units/labeling/tools accurately Carefully formulate explanations Calculate accurately and efficiently Express answers in terms of context Formulate and make use of definitions with others and their own reasoning.</td>
<td>Think aloud/Talk aloud Explicit instruction given through use of think aloud/talk aloud Guided Inquiry including teacher gives problem, students work together to solve problems, and debriefing time for sharing and comparing strategies Probing questions targeting content of study</td>
<td>What words can you use to explain what you did? What is another word for …? Your answer is (6). 6 what? Explain what you mean when you said that you counted them. How did you count them? How do you know that____?</td>
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<td><strong>7. Look for and make use of structure</strong></td>
<td>Look for, interpret, and identify patterns and structures Make connections to skills and strategies previously learned to solve new problems/tasks Reflect and recognize various structures in mathematics Breakdown complex problems into simpler, more manageable chunks</td>
<td>Be quiet and allow students to think aloud Facilitate learning by using open-ended questioning to assist students in exploration Careful selection of tasks that allow for students to make connections Allow time for student discussion and processing Foster persistence/stamina in problem solving Provide graphic organizers or record student responses strategically to allow students to discover patterns</td>
<td>What did you notice/see? What do you know that helps you…? Have you done something like this before?</td>
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<td><strong>8. Look for and express regularity in repeated reasoning</strong></td>
<td>Identify patterns and make generalizations Continually evaluate reasonableness of intermediate results Maintain oversight of the process</td>
<td>Provide rich and varied tasks that allow students to generalize relationships and methods, and build on prior mathematical knowledge Provide adequate time for exploration Provide time for dialogue and reflection Ask deliberate questions that enable students to reflect on their own thinking Create strategic and intentional check in points during student work time.</td>
<td>Will this always happen? How would we prove that …? What do you notice about …? What is happening in this situation? What would happen if…? Is there a rule for this?</td>
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Content Clusters

Excerpts from the Model Content Framework

Not all of the content in a given grade is emphasized equally in the standards. Some clusters require greater emphasis than the others based on the depth of the ideas, the time that they take to master, and/or their importance to future mathematics or the demands of college and career readiness. In addition, an intense focus on the most critical materials at each grade allows depth in learning, which is carried out through the Standards for Mathematical Practice.

To say some things have greater emphasis is NOT to say that anything in the standards can safely be neglected in instruction. Neglecting material will leave gaps in student skill and understanding and may leave students unprepared for the challenges of a later grade.

The following tables identify the Major Clusters, Additional Clusters, and Supporting Clusters for each of the grade levels.

- **Major Clusters** comprise the major work in the grade level.
- **Supporting Clusters** are designed to support and strengthen the major work.
- **Additional Clusters** may not connect tightly or explicitly to the major work; however, they are important to ensure that no gaps develop in later grades.

Kindergarten

<table>
<thead>
<tr>
<th>Key: ■ Major Clusters; ◆ Supporting Clusters; ○ Additional Clusters</th>
</tr>
</thead>
</table>

### Counting and Cardinality
- Know number names and the count sequence.
- Count to tell the number of objects.
- Compare numbers.

### Operations and Algebraic Thinking
- Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.

### Number and Operations in Base Ten
- Work with numbers 11-19 to gain foundations for place value.

### Measurement and Data
- Describe and compare measurable attributes.
- Classify objects and count the number of objects in categories.

### Geometry
- Identify and describe shapes.
- Analyze, compare, create, and compose shapes.
First Grade

**Operations and Algebraic Thinking**
- Represent and solve problems involving addition and subtraction.
- Understand and apply properties of operations and the relationship between addition and subtraction.
- Add and subtract within 20.
- Work with addition and subtraction equations.

**Number and Operations in Base Ten**
- Extending the counting sequence.
- Understand place value.
- Use place value understanding and properties of operations to add and subtract.

**Measurement and Data**
- Measure lengths indirectly and by iterating length units.
- Tell and write time.
- Represent and interpret data.

**Geometry**
- Reason with shapes and their attributes.

Second Grade

**Operations and Algebraic Thinking**
- Represent and solve problems involving addition and subtraction.
- Add and subtract within 20.
- Work with equal groups of objects to gain foundations for multiplication.

**Number and Operations in Base Ten**
- Understand place value.
- Use place value understanding and properties of operations to add and subtract.

**Measurement and Data**
- Measure and estimate lengths in standard units.
- Relate addition and subtraction to length.
- Work with time and money.
- Represent and interpret data.

**Geometry**
- Reason with shapes and their attributes.

Third Grade

**Operations and Algebraic Thinking**
- Represent and solve problems involving multiplication and division.
- Understand properties of multiplication and the relationship between multiplication and division.
- Multiply and divide within 100.
- Solve problems involving the four operations, and identify and explain patterns in arithmetic.

**Number and Operations in Base Ten**
- Use place value understanding and properties of operations to perform multi-digit arithmetic.

**Number and Operations—Fractions**
- Develop understanding of fractions as numbers.

**Measurement and Data**
- Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.
- Represent and interpret data.
- Geometric measurement: understand concepts of area and relate area to multiplication and to addition.
- Geometric measurement: recognize perimeter as an attribute of plane figures and distinguish between linear and area measures.

**Geometry**
- Reason with shapes and their attributes.
### Fourth Grade

**Operations and Algebraic Thinking**
- Use the four operations with whole numbers to solve problems.
- Gain familiarity with factors and multiples.
- Generate and analyze patterns.

**Number and Operations in Base Ten**
- Generalize place value understanding for multi-digit whole numbers.
- Use place value understanding and properties of operations to perform multi-digit arithmetic.

**Number and Operations—Fractions**
- Extend understanding of fraction equivalence and ordering.
- Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.
- Understand decimal notation for fractions, and compare decimal fractions.

**Measurement and Data**
- Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.
- Represent and interpret data.
- Geometric measurement: understand concepts of angle and measure angles.

**Geometry**
- Draw and identify lines and angles, and classify shapes by properties of their lines and angles.

### Fifth Grade

**Operations and Algebraic Thinking**
- Write and interpret numerical expressions.
- Analyze patterns and relationships.

**Number and Operations in Base Ten**
- Understand the place value system.
- Perform operations with multi-digit whole numbers and with decimals to hundredths.

**Number and Operations—Fractions**
- Use equivalent fractions as a strategy to add and subtract fractions.
- Apply and extend previous understandings of multiplication and division to multiply and divide fractions.

**Measurement and Data**
- Convert like measurement units within a given measurement system.
- Represent and interpret data.
- Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition.

**Geometry**
- Graph points on the coordinate plane to solve real-world and mathematical problems.
- Classify two-dimensional figures into categories based on their properties.

### Sixth Grade

**Ratios and Proportional Reasoning**
- Understand ratio concepts and use ratio reasoning to solve problems.

**The Number System**
- Apply and extend previous understandings of multiplication and division to divide fractions by fractions.
- Compute fluently with multi-digit numbers and find common factors and multiples.
- Apply and extend previous understandings of numbers to the system of rational numbers.

**Expressions and Equations**
- Apply and extend previous understandings of arithmetic to algebraic expressions.
- Reason about and solve one-variable equations and inequalities.
- Represent and analyze quantitative relationships between dependent and independent variables.

**Geometry**
- Solve real-world and mathematical problems involving area, surface area, and volume.

**Statistics and Probability**
- Develop understanding of statistical variability.
- Summarize and describe distributions.
### Seventh Grade

<table>
<thead>
<tr>
<th>Section</th>
<th>Details</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ratios and Proportional Reasoning</td>
<td>Analyze proportional relationships and use them to solve real-world and mathematical problems.</td>
</tr>
<tr>
<td>The Number System</td>
<td>Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.</td>
</tr>
<tr>
<td>Expressions and Equations</td>
<td>Use properties of operations to generate equivalent expressions.</td>
</tr>
<tr>
<td></td>
<td>Solve real-life and mathematical problems using numerical and algebraic expressions and equations.</td>
</tr>
<tr>
<td>Geometry</td>
<td>Draw, construct and describe geometrical figures and describe the relationships between them.</td>
</tr>
<tr>
<td></td>
<td>Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.</td>
</tr>
<tr>
<td>Statistics and Probability</td>
<td>Use random sampling to draw inferences about a population.</td>
</tr>
<tr>
<td></td>
<td>Draw informal comparative inferences about two populations.</td>
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<tr>
<td></td>
<td>Investigate chance processes and develop, use, and evaluate probability models.</td>
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### Eighth Grade

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<tr>
<td>The Number System</td>
<td>Know that there are numbers that are not rational, and approximate them by rational numbers.</td>
</tr>
<tr>
<td>Expressions and Equations</td>
<td>Work with radicals and integer exponents.</td>
</tr>
<tr>
<td></td>
<td>Understand the connections between proportional relationships, lines, and linear equations.</td>
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<tr>
<td></td>
<td>Analyze and solve linear equations and pairs of simultaneous linear equations.</td>
</tr>
<tr>
<td>Functions</td>
<td>Define, evaluate, and compare functions.</td>
</tr>
<tr>
<td></td>
<td>Use functions to model relationships between quantities.</td>
</tr>
<tr>
<td>Geometry</td>
<td>Understand congruence and similarity using physical models, transparencies, or geometry software.</td>
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<tr>
<td></td>
<td>Understand and apply the Pythagorean Theorem.</td>
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<tr>
<td></td>
<td>Solve real-world and mathematical problems involving volume of cylinders, cones and spheres.</td>
</tr>
<tr>
<td>Statistics and Probability</td>
<td>Investigate patterns of association in bivariate data.</td>
</tr>
</tbody>
</table>
Mathematical Tools and Representations
Excerpts from “How to Implement a Story of Units”, engage NY

The following are descriptions of mathematical tools that support students with developing a deep understanding of the math standards. For more information about specific instructional strategies utilizing these tools, visit: https://www.engageny.org

**Array and Area Models**

**Grade Level** 1 – 5

**Description**

An array is an arrangement of a set of objects organized into equal groups in rows and columns. Arrays help make counting easy. Counting by equal groups is more efficient than counting objects one by one. The ten-frame is an array used in Kindergarten. Students count objects in arrays in Kindergarten and Pre-Kindergarten. (PK.CC.4) The rectangular array is used to teach multiplication and leads to understanding area. (3.OA.3)

Arrays reinforce the meaning of multiplication as repeated addition (e.g., $3 \times 4 = 4 + 4 + 4$), and the two meanings of division—that $12 \div 3$ can indicate how many will be in each group if I make 3 equal groups and that it can also indicate how many groups I can make if I put 3 in each group. Further using arrays reinforces the relationship between multiplication and division.

**Base-Ten Blocks**

**Grade Level** K – 2

**Description**

Base-ten blocks (also referred to as Dienes blocks) include thousands “cubes,” hundreds “flats,” tens “rods,” and ones. Base-ten blocks are a proportional representation of units of ones, tens, hundreds, and thousands and are useful for developing place value understanding. This is a “pre-grouped” model for base-ten that allows for more efficient modeling of larger quantities through the thousands. However, because this place value model requires students to more abstractly consider the 10 to 1 relationship of the various blocks, care must be taken to ensure that students attend to the “ten-ness” of the pieces that are now traded rather than bundled or un-bundled.

Base-ten blocks are introduced after students have learned the value of hundreds, tens, and ones and have had repeated experiences with composing and decomposing groups of 10 ones or groups of 10 tens with bundles.
Bundles

Grade Level  
K – 2

Description

Bundles are discrete groupings of place value units (tens, hundreds, thousands), usually made by students/teachers placing a rubber band or chenille stem around straws, popsicle sticks, or coffee stirrers. Linking cubes may also be used in this fashion. Ten straws (or cubes) are bundled (or linked) into 1 unit of ten, 10 tens are bundled into 1 unit of a hundred, and so on. These student-made groupings provide the necessary conceptual foundation for children to be successful with pre-grouped, proportional, and non-proportional base-ten materials. (See Base-Ten Blocks and Number Disks.)

Understanding tens and ones is supported in Kindergarten as students learn to compose and decompose tens and ones by “bundling” and “unbundling” the materials. Numbers 11-19 are soon seen as 1 ten (a bundled set of 10 ones) and some extra ones.

By Grade 2, students expand their skill with and understanding of units by bundling units of ones, tens, and hundreds up to one thousand with sticks. These larger units are discrete and can be counted: “1 hundred, 2 hundred, 3 hundred, etc.” Bundles also help students extend their understanding of place value to 1000. (2.NBT.1) Repeated bundling experiences help students to internalize the pattern that 10 of one unit make 1 of the next larger unit. Expanded form, increased understanding of skip-counting (2.NBT.2), and fluency in counting larger numbers are all supported by the use of this model.

Bundles are also useful in developing conceptual understanding of renaming in addition and subtraction. The mat below shows 2 tens and 3 ones. To solve 23 – 9, one bundle of ten is “unbundled” to get 1 ten and 13 ones in order to take away 9 ones.

Money

Grade Level  
2

Description

Dollar bills (1s, 10s, and 100s) are non-proportional units that are used to develop place value understanding. That is, bills are an abstract representation of place value because their value is not proportionate to their size. Ten bills can have a value of $10 or $1000 but appear identical aside from their printed labels. Bills can be “traded” (e.g., 10 ten-dollar bills for 1 hundred-dollar bill) to help students learn equivalence of the two amounts.

As with other place value models, students can use bills to model numbers up to three digits, to read numbers formed with the bills, and to increase fluency in skip-counting by tens and hundreds.

The picture above shows that the arrangement of the $100s, $10s, and $1s can be counted in this manner:

The first frame, S: 100, 200, 300, 400, 500, 600, 700, 710, 720, 730.

The transition from a discrete unit of a “bundle” to proportional materials such as base-ten blocks to a non-proportional unit of a bill is a significant leap in a student’s place value learning trajectory.
Number Bond

Grade Level  K - 5

Description

The number bond is a pictorial representation of part-part-whole relationships and shows that within a part-whole relationship, smaller numbers (the parts) make up larger numbers (the whole). The number bond may be presented as shown, using smaller circles (or squares) for the parts to distinguish the part from the whole. As students become more comfortable using number bonds, they may be presented using the same size shape for parts and whole.

Number bonds of 10 have the greatest priority because students will use them for adding and subtracting across 10. Students move towards fluency in Grade 1 with numbers to 10 building on the foundation laid in Kindergarten. They learn to decompose numbers to ten with increasing fluency. (1.OA.6) Students learn the meaning of addition as “putting together” to find the whole or total and subtraction as “taking away” to find a part.

Notice in the diagrams below that the orientation of the number bond does not change its meaning and function. (6 + 2 = 8, 2 + 6 = 8, 8 - 6 = 2, 8 - 2 = 6)

Number Disks

Grade Level  2 - 5

Description

Number disks are non-proportional units used to further develop place value understanding. Like money, the value of the disk is determined by the value printed on it, not by its size. Number disks are used by students through Grade 5 when modeling algorithms and as a support for mental math with very large whole numbers. Whole number place value relationships modeled with the disks are easily generalized to decimal numbers and operations with decimals.
**Number Line**

**Grade Level**
K – 5

**Description**

The number line is used to develop a deeper understanding of whole number units, fraction units, measurement units, decimals, and negative numbers. Throughout Grades K-5, the number line models measuring units.

**Number Path**

**Grade Level**
PK – 1

**Description**

The number path can be thought of as a visual (pictorial) representation of the number tower (see description below) and is foundational to understanding and using the number line. It also serves as a visual representation of 1:1 correspondence and the concept of whole numbers (one number, one space, and each being equal in size). The color change at 5 helps to reinforce the 5 and 10 benchmarks. The number path also serves as an early precursor to measurement concepts and a support for cardinal counting. (If a student places 7 objects in each of the 7 spaces on the path, they must realize that there are 7 objects, not 10. Simply because the path goes up to 10 does not mean there are 10 objects.)

**Rekenrek**

**Grade Level**
PK – 5

**Description**

The Rekenrek has a 5 and 10 structure, with a color change at 5 (eliciting the visual effect of grouping 5 and grouping 10). The 20-bead Rekenrek consists of 2 rows of 10 beads, allowing students to see numbers to 10 either as a number line on one row or a ten-frame (5 beads on two rows). A 100-bead Rekenrek has 10 rows of 10 beads. Other names for the Rekenrek are “Calculating Frame,” “Slavonic Abacus,” “Arithmetic Rack,” or “Math Rack.”
Tape Diagram

Grade Level  1 – 5

Description

Rachel collected 58 seashells. Sam gave her 175 more. How many seashells did she have then?

Tape diagrams, also called bar models, are pictorial representations of relationships between quantities used to solve word problems. Students begin using tape diagrams in 1st grade, modeling simple word problems involving the four operations. It is common for students in 3rd grade to express that they don’t need the tape diagram to solve the problem. However, in Grades 4 and 5, students begin to appreciate the tape diagram as it enables students to solve increasingly more complex problems.

At the heart of a tape diagram is the idea of forming units. In fact, forming units to solve word problems is one of the most powerful examples of the unit theme and is particularly helpful for understanding fraction arithmetic.

The tape diagram provides an essential bridge to algebra and is often called “pictorial algebra.”

Like any tool, it is best introduced with simple examples and in small manageable steps so that students have time to reflect on the relationships they are drawing. For most students, structure is important. RDW (read, draw, write) is a process used for problem solving:

- Read a portion of the problem.
- Create or adjust a drawing to match what you’ve read. Label your drawing.
- Continue the process of reading and adjusting the drawing until the entire problem has been read and represented in the drawing.
- Write and solve an equation.
- Write a statement.

There are two basic forms of the tape diagram model. The first form is sometimes called the part-whole model; it uses bar segments placed end-to-end (Grade 3 Example below depicts this model), while the second form, sometimes called the comparison model, uses two or more bars stacked in rows that are typically left justified. (Grade 5 Example below depicts this model.)

Rather than talk to students about the 2 forms, simply model the most suitable form for a given problem and allow for flexibility in the students’ modeling. Over time, students will develop their own intuition for which model will work best for a given problem. It is helpful to ask students in a class, “Did anyone do it differently?” and allow students to see more than one way of modeling the problem, then perhaps ask, “Which way makes it easiest for you to visualize this problem?”

**Grade 3 Example**

Sarah baked 256 cookies. She sold some of them. 187 were left. How many did she sell?

256 cookies

| 187 left | sold |

256 - 187 = □

Sarah sold □ cookies.

**Grade 5 Example**

Sam has 1025 animal stickers. He has 3 times as many plant stickers as animal stickers. How many plant stickers does Sam have? How many stickers does Sam have altogether?

| Animal | 1025 |
| Plant | 1025 | 1025 | 1025 |

1 unit = 1025
3 units = 3075
2 units = 2050
4 units = 4100

1. He has 3075 plant stickers.
2. He has 4100 stickers altogether.
Ten-Frame

Grade Level		PK – 3

Description

A ten-frame is a 2 by 5 grid (array) used to develop an understanding of concepts such as 5-patterns, combinations to 10, and adding and subtracting within 20. The frame is filled beginning on the top row, left to right, then proceeding to the bottom row building left to right. This pattern of filling supports subitizing by building on the 5 benchmark, as well as providing a pattern for placing disks on place value mats in later grades. Concrete counters as well as pictorial dots may be used to represent quantities on the frame.

In Kindergarten and in early Grade 1 a double ten-frame can be used to establish early foundations of place value (e.g., 13 is 10 and 3 or 1 ten and 3 ones) and can also be used on place value mats to support learning to add double digit numbers with regrouping. The “completion of a unit” on the ten-frame in early grades empowers students in later grades to understand a “make 100 (or 1000)” strategy, to add 298 and 37 (i.e., 298 + 2 + 35), and to more fully understand addition and subtraction of measurements (e.g., 4 ft. 8 in. + 5 in).
Web-Based Resources

For Students:

- http://studyisland.com
- http://www.mathlearningcenter.org/web-apps/number-rack/
- http://www.ixl.com/math
- https://www.khanacademy.org/
- http://www.coolmath.com/

For Teachers:

- https://www.engageny.org/mathematics
- https://www.georgiastandards.org/Common-Core/Pages/Math-K-5.aspx
- http://illuminations.nctm.org/
- http://www.nctm.org/resources/content.aspx?id=538
- http://www.k-5mathteachingresources.com/
- https://pll.asu.edu

MATH MAKES YOUR LIFE ADD UP!
The Fibonacci sequence exhibits a certain numerical pattern which originated as the answer to an exercise in the first ever high school algebra text. This pattern turned out to have an interest and importance far beyond what its creator imagined. It can be used to model or describe an amazing variety of phenomena, in mathematics and science, art and nature. The mathematical ideas the Fibonacci sequence leads to, such as the golden ratio, spirals and self-similar curves, have long been appreciated for their charm and beauty, but no one can really explain why they are echoed so clearly in the world of art and nature.

The story began in Pisa, Italy in the year 1202. Leonardo Pisano Bigollo was a young man in his twenties, a member of an important trading family of Pisa. In his travels throughout the Middle East, he was captivated by the mathematical ideas that had come west from India through the Arabic countries. When he returned to Pisa he published these ideas in a book on mathematics called Liber Abaci, which became a landmark in Europe. Leonardo, who has since come to be known as Fibonacci, became the most celebrated mathematician of the Middle Ages. His book was a discourse on mathematical methods in commerce, but is now remembered mainly for two contributions, one obviously important at the time and one seemingly insignificant.

The important one: he brought to the attention of Europe the Hindu system for writing numbers. European tradesmen and scholars were still clinging to the use of the old Roman numerals; modern mathematics would have been impossible without this change to the Hindu system, which we call now Arabic notation, since it came west through Arabic lands.

The other: hidden away in a list of brain-teasers, Fibonacci posed the following question:

If a pair of rabbits is placed in an enclosed area, how many rabbits will be born there if we assume that every month a pair of rabbits produces another pair, and that rabbits begin to bear young two months after their birth?

This apparently innocent little question has as an answer a certain sequence of numbers, known now as the Fibonacci sequence, which has turned out to be one of the most interesting ever written down. It has been rediscovered in an astonishing variety of forms, in branches of mathematics way beyond simple arithmetic. Its method of development has led to far-reaching applications in mathematics and computer science.

But even more fascinating is the surprising appearance of Fibonacci numbers, and their relative ratios, in arenas far removed from the logical structure of mathematics: in Nature and in Art, in classical theories of beauty and proportion.

https://math.temple.edu/~reich/Fib/fibo.html